Perfect kernel of generalised Baumslag – Solitar groups

Setting: Γ a countable group

I/ Space of subgroups

1) Chabauty topology

- $\mathrm{Sub}(\Gamma) = \{\Lambda \subseteq \Gamma \mid \Lambda \text{ subgroup of } \Gamma\} \underset{\text{closed}}{\hookrightarrow} \{0,1\}^{\Gamma}$

Chabauty topology

- Basis of neighbourhoods: $O, I \underset{\text{finite}}{\subseteq} \Gamma$

$$\mathcal{V}(O, I) = \{\Lambda \leq \Gamma \mid I \subseteq \Lambda,\ O \cap \Lambda = \emptyset\}$$

basis of clopen sets

- Topology of the simple convergence:

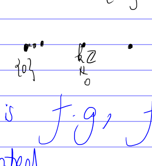
$\Lambda_n \xrightarrow[n\to+\infty]{} \Lambda$ iff $\left\{ \forall \lambda \in \Lambda\ (\notin),\ \exists n_0 \in \mathbb{N},\ \lambda \in \Lambda_n\ (\notin)\ \forall n \geq n_0 \right.$

- Ex: f.g subgroups are dense in $\mathrm{Sub}\,\Gamma$

$$\Lambda = \langle \lambda_i,\ i \in \mathbb{N} \rangle = \lim_{n\to+\infty} \langle \lambda_i,\ i \leq n \rangle$$

. In $\mathrm{Sub}\,\mathbb{Z}$, the only non isolated point is $\{0\}$

$\rightsquigarrow \mathrm{Sub}\,\mathbb{Z} \simeq \{0\} \cup \{\frac{1}{n}, n \in \mathbb{N}^*\}$



Prop: If Γ is f.g, finite index subgroups are isolated.

Proof: f.i subgroups are f.g.

So $\Lambda_n \xrightarrow[n\to+\infty]{} \underset{f.i}{\Lambda} \Rightarrow \exists n_0, \Lambda \leq \Lambda_n\ \forall n \geq n_0$

AQT$\boxtimes$

In practice, we will work with Schreier graphs. One has an identification

$$\left\{\begin{array}{l}\text{isomorphism classes of}\\ \text{pointed transitive Right actions}\\ (X, x_0) \overset{\alpha}{\curvearrowleft} \Gamma\end{array}\right\} \longleftrightarrow \mathrm{Sub}(\Gamma)$$

$$(X, x_0) \overset{\alpha}{\curvearrowleft} \Gamma \longmapsto \mathrm{Stab}_\alpha(x_0)$$

$$(\Lambda\backslash\Gamma, \Lambda) \curvearrowleft \Gamma \longleftarrow\!\shortmid \Lambda$$

Why is it useful? Assume $\Gamma = \langle \underset{\text{finite}}{S} \rangle$

Def: The Schreier graph of a pointed transitive right action $(X, x_0) \overset{\alpha}{\curvearrowleft} \Gamma$ is defined as follows:

- its set of vertices is X;
- $\forall x \in X, s \in S$, $\exists$ $x \xrightarrow{s} x\cdot s$

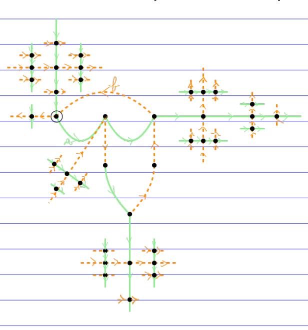
it is pointed at x_0.

Ex: If α is the right action of Γ on itself, $\mathrm{Sch}(\alpha) = \mathrm{Cay}_S(\Gamma)$.

Chabauty topology on the level of Schreier graphs: Two subgroups are close $\Longleftrightarrow$ their Schreier graphs are close

Ex: $\Gamma = \mathbb{F}_2 = \langle a, b \mid \emptyset \rangle$

$$\Lambda = \langle a^2 b,\ abab^3 \rangle$$



$\Rightarrow \Lambda$ is non isolated! $\boxtimes$

2) Perfect kernel

Thm (Cantor - Bendixson): There exists a unique decomposition $\mathrm{Sub}(\Gamma) = \underset{\substack{\text{closed}\\ \text{without isolated}\\ \text{point}}}{K(\Gamma)} \sqcup \underset{\text{countable}}{C}$

$\rightarrow K(\Gamma)$ is the largest closed subspace of $\mathrm{Sub}(\Gamma)$ without isolated point

$\rightarrow K(\Gamma)$ is the set of subgroups all of whose neighbourhoods are uncountable.

Ex: . Γ fg abelian group $\Rightarrow K(\Gamma) = \emptyset$

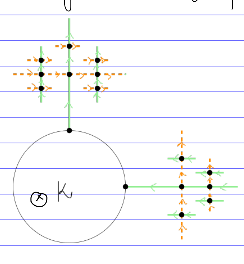
. known for abelian groups [Cornulier - Guyot - Pitsch 10]

. known for lamplighters [Bowen - Grigorchuk - Kravchenko 15]

Th: $K(\mathbb{F}_2) = \mathrm{Sub}_{[\infty]}(\mathbb{F}_2)$.

Proof: $\boxed{\subseteq}$ f.i subgroups are isolated.

$\boxed{\supseteq}$ it suffices to show that any f.g, ∞.i Λ is a non trivial limit of ∞.i subgroups.



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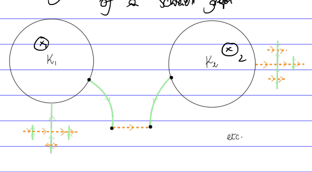
$\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$ by conjugation: action by homeomorphisms $\Rightarrow$ preserves the perfect kernel.

Def: $\underset{\text{countable group}}{\Gamma} \curvearrowright \underset{\text{top. space}}{X}$ is topologically transitive if for every $\mathcal{U}, V \underset{\neq\emptyset}{\subseteq} X$, $\exists g \in \Gamma$, $g\mathcal{U} \cap V \neq \emptyset$

Remark: If X is Polish, this is equivalent to the existence of a dense orbit.

Thm (folklore): $\mathbb{F}_2 \curvearrowright \underset{\mathrm{Sub}_{[\infty]}(\mathbb{F}_2)}{K(\mathbb{F}_2)}$ is topologically transitive.

Proof: change the base point of a Schreier graph $\Leftrightarrow$ conjugate!



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II/ Groups acting on trees

Thm [Azuelos - Gaboriau 23]: $\Gamma \curvearrowright T$ (minimally and irreducibly) such that the action is acylindrical.
($\exists R > 0$, every edge path of length $> R$ has finite stabilizer)

Then:

- $\mathrm{Sub}_{[\cdot\backslash T]\infty}(\Gamma) \subseteq K(\Gamma)$

Γ TT if Γ contains no non trivial finite normal subgroup.

Ex: $\rightarrow$ free groups!

$\rightarrow$ fg groups with ∞ many ends and without non trivial normal subgroup [Stallings]

In 2022, Carderi - Gaboriau - Le Maître - Stalder studied the perfect kernel of Baumslag - Solitar groups:

$$BS(m,n) = \langle t, b \mid t b^m t^{-1} = b^n \rangle.$$

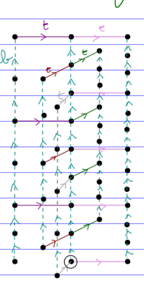
2) Generalized Baumslag - Solitar groups (GBS groups)

Def: A GBS group is a group Γ that acts cocompactly (i.e $\Gamma\backslash T$ finite) on an oriented tree T such that the vertex and edge stabilizers are $\simeq \mathbb{Z}$.

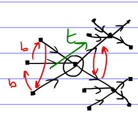
Typical examples of non acylindrical actions!

Ex: Baumslag - Solitar groups!

$BS(3,2) = \langle t, b \mid t b^3 t^{-1} = b^2 \rangle$



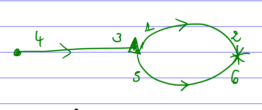
$\rightsquigarrow$ shrink the $\langle b \rangle$ - orbits



Bass - Serre theory: GBS groups are encoded by graphs of groups.

Def: A graph of groups $\mathcal{H}$ is a finite oriented graph all of whose half-edges are labelled by $\neq 0$ integers.

Ex:



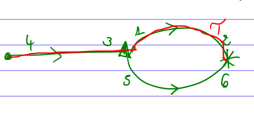
Construction of the corresponding GBS group

$\Rightarrow$ choose a spanning tree T;

generators $\left(\rightarrow \begin{cases} 1 \text{ generator } a_v \text{ for each } v \in \mathcal{V}(\mathcal{H}); \\ 1 \text{ generator } t_e \text{ for each } e \in \mathcal{E}(\mathcal{H}); \end{cases} \right.$

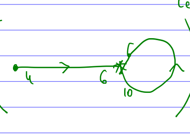
relations $\left(\rightarrow \begin{array}{l} .\ t_{\bar{e}} = t_e^{-1} \\ .\ \forall\ \ast \xrightarrow[k\qquad\ell]{e} \blacktriangle,\ t_e a_\ast^k t_e^{-1} = a_\blacktriangle^\ell \\ .\ t_e = 1 \quad \forall e \in \mathcal{E}(T) \end{array} \right.$

Ex: $\mathcal{H} =$



$$\Gamma \underset{(\text{not.})}{=} \pi_1(\mathcal{H}) = \left\langle a_\bullet, a_\blacktriangle, a_\ast, t_e \,\middle|\, \begin{array}{l} a_\bullet^4 = a_\blacktriangle^3 \\ a_\blacktriangle^1 = a_\ast^2 \\ t_e a_\blacktriangle^5 t_e^{-1} = a_\ast^6 \end{array} \right\rangle$$

$\simeq \pi_1 ($



$)$

$\Rightarrow \mathcal{H}$ not unique in general.

From now we assume that Γ is a GBS group defined by a reduced graph of groups $\mathcal{H}$.
(the only edges that have a label ± 1 are loops)

T associated Bass - Serre tree.

Thm 1: (B. '24): $K(\Gamma) = \mathrm{Sub}_{[\cdot\backslash T]\infty}(\Gamma)$

$= \begin{cases} \mathrm{Sub}_{[\infty]}(\Gamma) & \text{if } \Gamma \text{ non unimodular} \\ \pi^{-1}(\mathrm{Sub}_{[\infty]}(\Gamma/N)) & \text{if } \exists N \trianglelefteq \Gamma,\ N \simeq \mathbb{Z} \end{cases}$

$\rightsquigarrow$ Similar to AG 23 which dealt with acylindrical actions.

Thm. 2 (B. '24): $\exists$ countably infinite invariant partition $K(\Gamma) = \bigsqcup_{n \in \mathbb{N}^* \sqcup \{\infty\}} K_n$ s.t:

$\rightarrow K_n$ open $\forall n \in \mathbb{N}^*$ (and also closed iff Γ is unimodular)

$\rightarrow K_\infty$ closed.

$\rightarrow \Gamma \curvearrowright K_n$ is TT $\forall n \in \mathbb{N}^* \sqcup \{\infty\}$.

$\rightsquigarrow$ prevents $\Gamma \curvearrowright K(\Gamma)$ from being TT!

3) Ideas of the proofs

- As for $\mathbb{F}_2$: "cut and paste" Schreier graphs

⚠ $\bullet, \ast$ vertices of $\mathcal{H}$

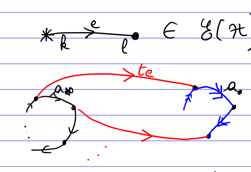
$1\ \langle a_\bullet \rangle$ is $\infty \Longleftrightarrow$ all $\langle a_\ast \rangle$ - orbits are ∞

$\rightsquigarrow$ when we cut Schreier graphs, we keep the whole orbits of vertices

- To make the decomposition explicit: find a quantity which is invariant under conjugation

Idea: $\ast \xrightarrow[k\qquad\ell]{e} \bullet \in \mathcal{E}(\mathcal{H})$

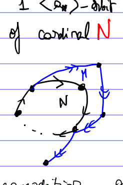
$e \notin T$



$\Rightarrow \dfrac{N}{N \wedge k} = \dfrac{M}{M \wedge \ell}$

1 $\langle a_\ast \rangle$ - orbit of cardinal N — 1 $\langle a_\bullet \rangle$ - orbit of cardinal M

$e \in T$

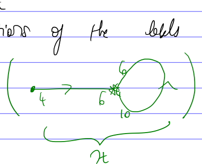


$\Rightarrow \dfrac{N}{N \wedge k} = \dfrac{M}{M \wedge \ell}$

The decomposition of Thm. 2 is computable and depends on

- the skeleton of $\mathcal{H}$
- the p-adic valuations of the labels of $\mathcal{H}$.

Ex: $\Gamma = \pi_1 \Big(\underbrace{\qquad}_{\mathcal{H}} \Big)$



choose a vertex ($\bullet$ for example)

Define $\mathrm{Ph}_{\mathcal{H},\bullet} : \mathbb{N}^* \sqcup \{\infty\} \longrightarrow \mathbb{N}^* \sqcup \{\infty\}$

$\infty \longmapsto \infty$

$N \longmapsto \displaystyle\prod_{\substack{p \in \mathcal{P} \mid \\ |N|_p > |4|_p, \\ |N|_p > |4|_p - |6|_p + |10|_p, \\ |10|_p = |6|_p}} p^{|N|_p}$

$\left(\displaystyle\prod_{\substack{p \in \mathcal{P} \mid \\ p \neq 2,3,5}} p^{|N|_p} \right) \times \begin{cases} p^{|N|_2} & \text{if } |N|_2 > 2 \\ 1 & \text{otherwise} \end{cases}$